Effect of Nonlinear Characteristics on Temperature Distribution in a Rocket Grain

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Introduction

THE internal passages of a solid-propellant motor possess, in general, an extremely complicated geometry. These internal passages are three-dimensional in character, having shapes that are usually dictated by internal ballistics considerations. Very frequently these internal grain perforations are two dimensional in nature over a very substantial portion of the total length of the motor.[1] It is then reasonable to construct a mathematical model which consists of an infinitely long circular cylinder with a star-shaped perforation.

This Note deals with the approximate, analytic determination of the steady-state temperature field in such a two-dimensional system, taking into account the variation of the heat conduction coefficient with temperature.

It is felt that this is a rather realistic consideration since solid-propellant rocket grains are constituted of physically nonlinear materials. On the other hand, thermal stresses will be affected by nonlinearities of this type.

Approximate Analysis of the Problem

The approach followed herein consists of a straightforward extension of the methodology presented in Ref. 2.

Since the problem under consideration is governed by the nonlinear, steady-state, two-dimensional Fourier heat equation,

$$\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[k(T)\frac{\partial T}{\partial y}\right]=0 \tag{1}$$

it is convenient to make a change in temperature scale through the relation

$$T=T_i\tau \tag{2}$$

where T_i is the temperature of the inner boundary, and T_e is the temperature at the outer boundary.

Substituting Eq. (2) in Eq. (1) results in

$$\frac{\partial}{\partial x}\left(K\frac{\partial \tau}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial \tau}{\partial y}\right)=0 \tag{3}$$

where $k(T)=K(\tau)$ and τ is subject to the boundary conditions

$$\tau_i=1, \qquad \tau_e=T_e/T_i$$

Now introduce the change in variable[3]

$$\phi(\tau)=\frac{1}{K_i}\int_1^\tau K(\tau)\,\mathrm{d}\tau \tag{4}$$

where $K_i=K(\tau_i)$.

Since for many engineering materials the variation of the heat conduction coefficient with respect to temperature is

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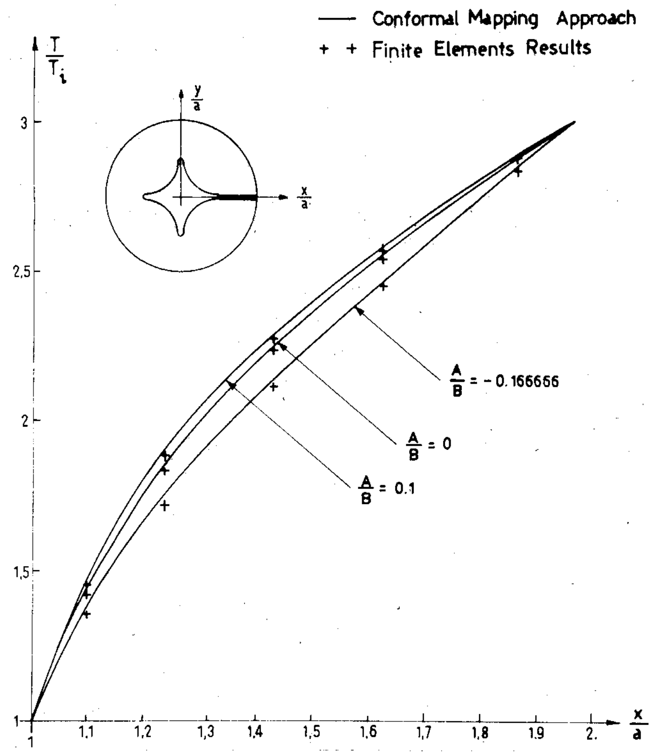

Fig. 1 Temperature distribution ($\alpha=0$ deg).

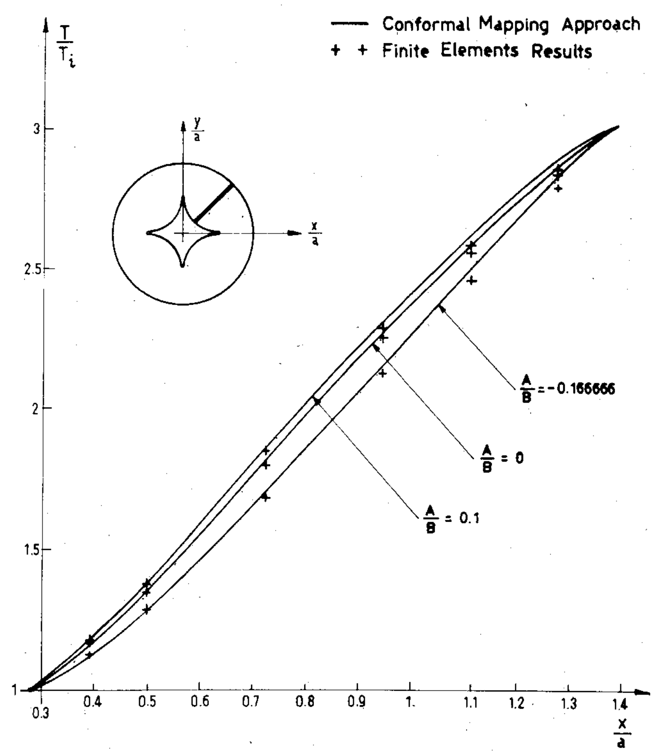

Fig. 2 Temperature distribution ($\alpha=45$ deg).

approximately linear, it will be assumed here that

$$k(T)=aT+b=aT_i\tau+b=K(\tau) \tag{5}$$

Accordingly,

$$K(\tau)/K_i=aT_i/K_i+b/K_i=A\tau+B \tag{6}$$

where

$$A=aT_i/K_i, \qquad B=b/K_i$$

Substituting Eqs. (4) and (6) in Eq. (3) one obtains

$$\frac{\partial^2}{\partial x^2}\left(\frac{\phi}{B}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\phi}{B}\right)=0 \tag{7}$$

Finally, from Eqs. (6) and (4) one gets

$$\frac{\phi(\tau)}{B}=\frac{A}{2B}\tau^2+\tau-\left(\frac{A}{2B}+1\right) \tag{8}$$

The governing differential system is then constituted by Eq. (7) and the appropriate boundary conditions which, making use of Eq. (8), become

$$\phi(1)/B=0$$

$$\frac{\phi(\tau_e)}{B}=\phi_e=-\frac{A}{2B}\tau_e^2+\tau_e-\left(\frac{A}{2B}+1\right) \tag{9}$$

Once the differential system is solved one is able to determine τ using Eq. (9) since

$$\tau=\frac{B}{A}\left[-1+\sqrt{1+\frac{2A}{B}\left(\frac{A}{2B}+1+\frac{\phi(\tau)}{B}\right)}\right] \tag{10}$$

The determination of the solution of Eq. (7) will now be discussed. This equation is the classical Laplace equation and since it is invariant under transformation, it is convenient to make use of the conformal mapping method in order to transform the given domain (circular with a star-shaped perforation) onto a simpler one, say a circular annulus.

The functional relation which accomplishes such a transformation is a Laurent-type series of the form[1]

$$z=\sum_{-\infty}^{\infty}\alpha_i\zeta^i;\quad \zeta=r_e^{i\theta};\quad 1\le r\le r_e \tag{11}$$

Reference 1 contains a discussion of methods which allow for the determination of Eq. (11).

Now expressing Eq. (7) in terms of the (r,θ) variables in the ζ plane one gets

$$\frac{d^2}{dr^2}\left(\frac{\phi}{B}\right)+\frac{1}{r}\frac{d}{dr}\left(\frac{\phi}{B}\right)=0 \tag{12}$$

The solution of Eq. (12) is simply

$$\frac{\phi}{B}=\phi_e\frac{\ln r}{\ln r_e} \tag{13}$$

Summarizing, the dimensionless temperature variable T/T_i is defined by Eqs. (2), (10), (11), and (13).

Numerical Results

In order to simplify the calculations, one considers a grain cross section mapped by the functional relation[4]

$$z=a(0.7789\zeta+0.2965\zeta^{-3}-0.0789\zeta^{-7}+0.0034\zeta^{-11}) \tag{14}$$

where a is a scale factor. Making $r_e=2.50$ one obtains a grain cross section with an outer radius $R_e=1.966a$ (the maximum deviations with respect to a perfect circle are less than 1%). It is important to point out that finding the mapping function is, in general, a very difficult problem since it reduces to the solution of two coupled integral equations[1] and then determining the coefficients of the truncated Laurent-type expansion.

Figures 1 and 2 depict dimensionless temperature variations T/T_i for a hypothetical situation where $T_e/T_i=3$. Three situations are considered: 1) $A/B=0.10$, 2) $A/B=-0.1666$, and 3) $A/B=0$.

Also shown in Figs. 1 and 2 are the dimensionless temperature values obtained by means of a finite element code. The agreement is quite good from an engineering viewpoint in all cases considered.

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